A six-dimensional $(Z_2)^3$ symmetric model with warped physical space

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Abstract

The Randall–Sundrum model is studied in six dimension with AdS_4 or dS_4 metric in the physical four-dimensional space. Two solutions are found, one with induced five-dimensional gravity terms added to the induced cosmological constant terms. We study the graviton modes in both solutions by transforming the mass eigenvalue equation to a Schrodinger equation with a volcano potential. The spectrum of gravitational excitations depends on the input parameters of the theory, the six-dimensional and the effective four-dimensional cosmological constants. The model gives a physically acceptable spectrum if the four-dimensional cosmological constant is sufficiently small.

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1. Introduction

Recent intensive study of string theory revived interest in gravitational models with more than four spacetime dimensions. Models with extra (compactified or infinite) dimension, confining particles on 3-branes have been in the forefront of research in particle physics in the past few years [1–4]. These models have features that, unlike that of traditional string theories, may make the observation of extra dimensions possible in the near future. One of the most successful models proposed by Randall and Sundrum (RS) [3, 4] is based on an orbifold solution of the five-dimensional Einstein equation in AdS space, with a 3-brane fixed at the zero of the fifth coordinate, u (and possibly at another value of u, as well), around which the metric has a Z_2 symmetry. This model may provide a solution to the hierarchy problem of particle physics. It confines gravitons to the neighborhood of a brane.

The RS model with infinite extra dimensions [4] was extended to $D > 5$ dimensions by Arkani-Hamed *et al* [5]. The model has intersecting $(D − 2)$-branes with gravity localized at the intersection. At long distances, along the intersection, Newton gravity is recovered.

With an appropriately chosen curvature Newton's force is modified only at sub-millimeter distances. The cosmological constants were fine tuned to result in a Minkowski space at the intersection of the (D-2)-branes.

The original RS model is required to satisfy a fine tuning condition if we demand that the 3-brane of our world has Minkowski metric. However, in our world the cosmological constant is probably nonzero and could have had a large value in the past. Thus, it is of interest to investigate the generalization of the RS model when the physical four-dimensional subspace is not Minkowski (M_4), but AdS_4 or dS_4. Such a model has been investigated in five dimensions [6–9]. In the horospheric coordinates, the space is infinite for AdS_5, while it is finite for dS_5. For distances much smaller than the four-dimensional curvature radius and much larger than the five-dimensional curvature radius Newtonian gravity is reproduced.

The aim of the present paper is to extend the $D = 6$ case of [6–9] to a warped four-dimensional subspace. Just like [5], we require $Z_2 \times Z_2$ orbifold symmetry in the two extra dimensions. In addition, for simplicity, we also impose Z_2 symmetry under the exchange of the two extra coordinates. In a previous work Chodos and Poppitz [10] discussed a 3-brane embedded in a six-dimensional bulk using a different type of ansatz. Their solution also incorporated a nonvanishing cosmological constant on the physical 3-brane but the symmetry of their ansatz is $AdS_4 \times E(2)$.

Kaloper also found a $D = 6$ solution with dS or AdS geometries on the physical 3-brane [11]. His solution uses two 4-branes intersecting at an angle different from the right angle. The model has $Z_2 \times Z_2$ symmetry. It is asymmetric in the exchange of the two extra dimensions. Upon the analysis of small fluctuations no confined graviton mode was found in the case of AdS geometry on the physical 3-brane.

In our analysis we have found two new solutions, both with two intersecting perpendicular 4-branes in the action. They are symmetric to the exchange of the two extra coordinates. These solutions will be discussed in the next section. In section 3 we investigate small oscillations around the global solutions. In a subset of solutions that includes the ground state we are able to transform the mass eigenvalue equation to a Schrodinger form, with a now well-known volcano potential. We also find the spectrum of Kaluza–Klein modes. In section 4 we conclude our paper. Two appendices contain some details of the calculations.

2. Solution of the Einstein equation

In six dimensions, when the four-dimensional physical space is Minkowski, a solution can be found for the orbifold problem [5], just like in five dimensions, using conformal coordinates. One cannot however transform the coordinates to a horospheric coordinate system that has been used to solve the five-dimensional RS model without fine tuning [8, 9].

Unfortunately, in six (or higher) dimensions, when the four-dimesional space is warped, the metric cannot be brought either to the horospheric, or to the conformal form. Each of these coordinate systems would only contain a single symmetric function of the two extra coordinates, while, as we will soon see, the metric components depend on at least one non-symmetric function of these variables. In fact, using appropriate coordinate transformations the most general $Z_2 \otimes Z_2 \otimes Z_2$ symmetric ansatz for a six-dimensional metric which is AdS (dS) in the four-dimensional subspace (AdS_4 or dS_4) that, with a further gauge transformation, can be brought to the form

$$ds^2 = \gamma_{AB}\, dx^A\, dx^B \equiv \Omega^2[g_{\mu\nu}\, dx^\mu\, dx^\nu + g(u, v)\, du^2 + g(v, u)\, dv^2], \tag{1}$$

where $g_{\mu\nu}$ is a four-dimensional metric satisfying the Einstein equation with a four-dimensional cosmological constant λ, which may be positive (dS_4 space), zero (Minkowski space) or

negative (AdS$_4$ space). Finally, Ω is a symmetric function that can be brought to the form of the conformal factor for M$_4$ space [5],

$$\Omega = \frac{1}{[1 + a(|u| + |v|)]^2}. \tag{2}$$

With this choice, for M_4 space, the metric would reduce to the form of [5] and $g[u, v] = 1$.

The only unknown function in the metric, $g(u, v)$, has no symmetry property for the exchange of the two coordinates. As we will see below, the global solution of the Einstein equation for function $g(u, v)$ contains a single integration constant. This constant will be fixed later using the Israel junction conditions.

2.1. Global solution of the Einstein equation

The nonzero components of the Einstein tensor are $\mathcal{G}_{\mu\mu}$ with $\mu = t, x, y, z, \mathcal{G}_{uu}, \mathcal{G}_{vv}$ and $\mathcal{G}_{uv}$. The four components, $\mathcal{G}_{\mu\mu}$ provide identical second-order partial differential equations in variables u and v. It is easy to show that these equations follow, due to a Jacobi identity, from the rest of the equations. Thus, the bulk equations $\mathcal{G}_{uu} = -g_{uu}\Lambda_6$ and $\mathcal{G}_{uv} = 0$ are sufficient to find a unique (up to an integration constant) solution. The equation $\mathcal{G}_{vv} = -g_{vv}\Lambda_6$ follows from the first equation using the replacement $u \leftrightarrow v$. These equations contain only first-order derivatives of the unknown function $g(u, v)$.

To simplify notations we will use variables u and v rather than $|u|$ and $|v|$. We may assume for the purpose of finding bulk (six-dimensional) solutions that $u, v > 0$. The role of $Z_2 \times Z_2$ symmetry is important only when one solves the junction conditions that we will discuss after finding the global solution of the Einstein equation.

The details of the solution are presented in appendix A. We just recapitulate the results. The solution of the system of equations is given in the form

$$g(u, v) = \frac{1}{S(\sigma) + A(\delta, \sigma)}, \tag{3}$$

where we defined $\sigma = u + v$ and $\delta = u - v$. $A(\delta, \sigma)$ is an odd function of δ. $S(\sigma)$ is given by

$$S(\sigma) = \frac{1}{a^2} \left[\frac{(1 + a\sigma)^2 \lambda}{6} - \frac{\Lambda_6}{20} + C(1 + a\sigma)^5 \right], \tag{4}$$

where C is a yet undetermined integration constant.

The Taylor series of $A(\delta, \sigma)$ in δ is

$$A(\delta, \sigma) = \sum_{k=0}^{\infty} \delta^{2k+1} \alpha_k(\sigma), \tag{5}$$

where

$$\alpha_0(\sigma) = S'(\sigma), \tag{6}$$

and α_k satisfies the recursion relation

$$\alpha_k(\sigma) = -\frac{1}{S(\sigma)(2k+1)} \sum_{m=0}^{k-1} \alpha'_m(\sigma)\alpha_{k-1-m}(\sigma). \tag{7}$$

The recursion relation (7) can be used to generate α_k in an arbitrary order. The first few α_k are listed in appendix A.

Note now that $g(u, v)$ is unique, except for the choice of the scale a and the integration constant C. One of these constants can be fixed by setting the scale of variables u and v. The other will be fixed by the junction conditions to be discussed later.

As usual, the scale is set by demanding that the metric is Minkowski at $u = v = 0$. As $A(0, 0) = 0$ we find the following value for the conformal scale from (4):

$$a = \sqrt{\frac{\lambda}{6} - \frac{\Lambda_6}{20} + C}. \tag{8}$$

Note that a solution may exist for both positive and negative cosmological constants, as long as $\frac{\lambda}{6} - \frac{\Lambda_6}{20} + C > 0$.

2.2. Israel junction conditions

Due to the requirement of $Z_2 \times Z_2$ symmetry the derivatives of the components of the metric tensor are not continuous at $u = 0$ and $v = 0$. This discontinuity will generate junction terms in the Einstein tensor, localized on the $u = 0$ and $v = 0$ planes. These junction terms are

$$\Delta G^{(6)}_{\mu\mu} = -g_{\mu\mu}\delta(v)\left[\frac{8a(1+au)}{g(0,u)} - \frac{g_{,v}(u,v)|_{v=0}(1+au)^2}{g(0,u)g(u,0)}\right] + (u \leftrightarrow v),$$

$$\Delta G^{(6)}_{uu} = -\delta(v)g_{uu}\frac{8a(1+au)}{g(0,u)} \tag{9}$$

$$\Delta G^{(6)}_{vv} = -\delta(u)g_{uu}\frac{8a(1+av)}{g(0,v)}.$$

To satisfy the Einstein equations at $u = 0$ and $v = 0$ two 4-brane contributions are required to cancel (9). Brane contributions come from the variation of $u = 0$ and $v = 0$ 4-brane terms such as a tension term (i.e. a five-dimensional cosmological constant term) and possibly of dynamical terms. As we will see later the junction conditions will impose three u and v independent constraints on the constants of the theory. One of these can be used to fix the integration constant C. Then the brane contributions must in general contain two additional constants. One of these is the cosmological constant, Λ_5. We choose the other constant as the five-dimensional gravitational constant, G_5. Thus, we propose to add the following terms to the action:

$$S_5 = \frac{1}{8\pi G_5}\left[\int d^4x\,du\sqrt{-g}R^{(5)} + \Lambda_5 \int d^4x\,du\sqrt{-g} + (u \leftrightarrow v)\right], \tag{10}$$

where $R^{(5)}$ is the five-dimensional Ricci scalar constructed from the induced metric on the $v = 0$ or $u = 0$ brane. It is known that adding a four-dimensional gravity term to the five-dimensional action changes predictions for post-Newtonian effects, contradicting observations [12–14]. The question, whether a similar problem arises about our solution with induced five-dimensional gravity will be taken up in a future publication.

The variation of (10) gives

$$\mathcal{G}^{(5)}_{\mu\mu} + g_{\mu\mu}\Lambda_5 = g_{\mu\mu}\left[\Lambda_5 - \lambda(1+au)^2 + \frac{6a^2}{g(u,0)} + \frac{3a}{2}(1+au)\frac{g_{,u}(u,0)}{g^2(u,0)}\right],$$

$$\mathcal{G}^{(5)}_{uu} + g_{uu}\Lambda_5 = g_{uu}\left[\Lambda_5 - 2\lambda(1+au)^2 + \frac{6a^2}{g(u,0)}\right]. \tag{11}$$

There are similar contributions on the $u = 0$ brane, obtained from (11) by the exchange $(u \leftrightarrow v)$. One of the advantages of maintaining the $u \leftrightarrow v$ symmetry is that one needs to deal with two junction conditions rather than four.

It is easy to see that when we substitute our solution $g(u, v)$ into (9) and (11) multiplied by $\delta(v)$ the sum of these contributions does not vanish at all values of u. This alone does not mean that the six-dimensional junction term cannot be canceled by the brane terms of (10). The gauge of the junction contributions and of the brane contributions may be misaligned. In the following section we will show that this is indeed the case.

   Another way to satisfy the junction conditions would be to add further invariants to the action, such as Lovelock terms, but one would need an infinite number of such terms in the Lagrangian to have a chance to satisfy infinitely many constraints imposed by the junction conditions.

2.3. Coordinate transformation

As the six-dimensional action is gauge invariant, we can apply a gauge transformation to its contribution to the junction condition. We require that the gauge transformations satisfy the following conditions:

  (i) The $(u \leftrightarrow v)$ exchange symmetry is maintained. By this requirement we reduce the number of junction equations to be satisfied from four to two.
 (ii) We require that the transformed coordinates are such that u' and v' vanish when u and v vanish, respectively. This constraint is required by the junction equations. It implies that the position of the branes is unchanged by the gauge transformation.
(iii) We will maintain the scale of the coordinates at $u = v = 0$, by requiring that the components of the gauge transformed metric tensor $\tilde{g}_{uu}$ and $\tilde{g}_{vv}$ also tend to one when $u, v \to 0$.

   These requirements lead to the following form of coordinate transformations:

$$u \to u' = h(u, v) \qquad v \to v' = h(v, u), \tag{12}$$

where $h(u, v)$ is a C^2 function, $h(0, v) = 0$, $h_{,u}(u, v)|_{u=v=0} = 1$.

   We will show in appendix B that the two junction conditions can be satisfied by an appropriate choice of $h(u, v)$. While that result is important, the physical consequences of our solution can be explored without the detailed knowledge of the form of the function $h(u, v)$ provided $h(0, 0) = 0$ and $h_{,u}(u, v)|_{u=v=0} = 1$.

   When one writes the junction conditions one must take care of the fact that the action is varied with respect to g_{AB}, rather than $\tilde{g}_{AB}$. In other words, certain linear combinations of (9) enter the junction conditions. Denoting these appropriate linear combinations by $\Delta R^{(6)}_{AB}$ the junction conditions at $v = 0$ are

$$\frac{\sqrt{-\tilde{g}}}{G_6} \Delta_v R^{(6)}_{AB} + \frac{\sqrt{-g}}{G_5} R^{(5)}_{AB} \delta(v) = 0. \tag{13}$$

One obtains two independent junction conditions, one for choosing $A = B = \mu$ and other for $A = B = u$ (see appendix B). The junction contributions are functionals of two independent single variable functions, $f_1(u) = h(u, 0)$ and $f_2(u) = h_{,v}(u, v)|_{v=0}$, in an algebraic manner. The only constraints on these functions are $f_1(0) = 0$ and $f_1'(0) = f_2(0) = 1$. We must satisfy these three independent constraints. Once these constraints are satisfied one can always find two functions, $f_i(u)$, which solve the two junction conditions. The function $f_2(u)$ enters the junction conditions in a very simple manner: its third power is a multiplier of the six-dimensional junction contributions. Taking the two junction conditions at $u = 0$ provides two constraints on the constants of the model. Taking the ratio of the two junction conditions eliminates $f_2(u)$ from the resulting condition. Taking a derivative of the resulting equation with respect to u, at $u = 0$, provides a third constraint. The function $h(u, v)$ enters each of the three constraints only through the combinations $f_1(0) = 0$, $f_1'(0)$ and $f_2(0)$, all fixed by our constraints on the gauge transformation. All other derivatives of the junction conditions taken at $u = 0$ contain unconstrained derivatives of $h(u, v)$ that are determined by these very equations.

It is clear now why, in general, we need the two terms in (10) (cosmological constant and gravity). Recall that besides the coupling constants G_5 and Λ_5 we still have one undetermined integration constant, C (or, alternatively the conformal scale, a, which is related to C through (8)). The three equations obtained from the junction conditions are just sufficient to fix these three constants. In fact, one could replace the five-dimensional scalar curvature term of the Lagrangian by another gauge-invariant term, such as a Lovelock term, but the scalar curvature term is the simplest choice.

The two junction conditions taken at $u = 0$ provide the following equations:

$$0 = -\frac{8a}{G_6} - \frac{1}{G_5}\left(9a^2 - \frac{1}{2}\lambda - \Lambda_5 + \frac{3}{4}\Lambda_6\right) \tag{14}$$

$$0 = -\frac{8a}{G_6} - \frac{1}{G_5}(2\lambda - 6a^2 - \Lambda_5), \tag{15}$$

where we used the expression, obtained from our solution,

$$ag_{,u}(u,0)|_{u=0} = \lambda - 10a^2 - \tfrac{1}{2}\Lambda_6. \tag{16}$$

The set of equations (14) and (15) have two solutions.

(i) Subtracting the two equations from each other implies

$$a^2 = \frac{\lambda}{6} - \frac{\Lambda_6}{20}, \tag{17}$$

or in other words, the integration constant $C = 0$. Then we also have

$$\Lambda_5 = \frac{8aG_5}{G_6} + \lambda + \frac{3}{10}\Lambda_6. \tag{18}$$

The third constraint equation, to be discussed below will also determine the five-dimensional gravitational constant, G_5.

(ii) Note that if we take the limit $G_5, \Lambda_5 \to \infty$ (no scalar curvature term on the brane) such that $\tilde{\Lambda}_5 = \Lambda_5/8\pi G_5$ is finite then (14) and (15) become identical, providing an expression for $\tilde{\Lambda}_5$

$$\tilde{\Lambda}_5 = \frac{a}{\pi G_6}. \tag{19}$$

The third constraint equations will fix the yet undetermined a (or, alternatively, fix C).

Now for solution 1, the mere presence of the dynamical term on the $v = 0$ and $u = 0$ branes, combined with the fact that the metric tensors depend on $|u|$ and $|v|$, respectively, requires the existence of a non-vanishing tension on the $u = v = 0$ 3-branes, just like in the five-dimensional Randall–Sundrum solution. We denote the tension by Λ_4. We obtain

$$\Lambda_4 = -6\lambda. \tag{20}$$

If $\lambda = 0$ (non-warped space) the contribution on the 3-brane vanishes and the solution reduces to that of [5].

Let us consider now the third and final constraint equation. As we indicated earlier it is obtained from the derivative of the ratio of the two junction conditions, eliminating $h_{,v}(v,u)|_{v=0}$ from the equation. To reduce the obtained equations to a constraint on our constants we need to calculate derivatives of $g(u,v)$ at $u = v = 0$. These can be determined either using the explicit form of $g(u,v)$ from (3) or directly from taking derivatives of (14) and (15) and setting $u = v = 0$. To simplify matters we write the third constraint separately for the first and second solution. For the first solution (G_5 = finite) we obtain, using (17)

$$1700\Lambda_5^2 - 40\Lambda_5(80 + 27\Lambda_6) + 3\left(500\lambda^2 + 340\lambda\Lambda_6 + 57\Lambda_6^2\right) = 0. \tag{21}$$

Table 1. Summary of parameters characterizing the two solutions.

Solution	a	C	G_5	$\Lambda_5\ (\tilde{\Lambda}_5)$	Λ_4	$S(\sigma)$
Solution I	$\pm\sqrt{\frac{\lambda}{6} - \frac{\Lambda_6}{20}}$	0	$-\frac{3aG_6}{34}$	$-10\lambda - \frac{19}{5}\Lambda_6$	-6λ	$-rw^2 + 1 + r$
Solution II	$\pm\sqrt{\frac{11\lambda}{100} - \frac{\Lambda_6}{20}}$	$-\frac{17\lambda}{300}$	∞	$\frac{a}{\pi G_6}$	0	$1 + r\left(\frac{33+17w^5}{50} - w^2\right)$

For the second solution ($G_5 = \infty$), after substituting (19), we obtain for the third constraint

$$100a^2 - 11\lambda + 5\Lambda_6 = 0. \tag{22}$$

These equations can be used to calculate the rest of the constants. All the constants of the theory are determined by the parameters of the six-dimensional world, G_6 and Λ_6 and of the physical cosmological constant, λ. The value of parameters found for the two solutions are given in table 1, where we use the notations $r = -\lambda/6a^2$ and $w = 1 + a\sigma$.

We can see from table 1 that in fact, taking into account the two possible signs for a, we have four independent solutions, rather than two. Note that at a special choice of the parameter $r, r = -38/41$ the combination $50\lambda + 19\Lambda_6 = 0$ and $\Lambda_5 = 0$. Then the geometry on the intersecting branes is asymptotically Minkowski.

2.4. The range of variables u and v

The range of variables u and v plays an important role in the investigation of small deformations of the above solutions. The linearized deformations determine the spectrum of gravitational excitations and the nature of the gravitational force among massive objects.

Form (3) of the solution implies that zeros of $S(\sigma) \pm A(\delta, \sigma)$ (for both signs) constitute boundaries in the (u, v) space. Suppose the equation of such a boundary is on the curve $\delta = \delta(\sigma)$. Then $\delta(\sigma)$ satisfies

$$[g(u, v)]^{-1} = S(\sigma) + A(\delta(\sigma), \sigma) = 0. \tag{23}$$

This equation can easily be solved. Consider that whenever (23) is satisfied, unless $\delta = 0, [g(v, u)]^{-1} \neq 0$. Then the equation $\mathcal{G}_{uv} = 0$, (A.2) of appendix A implies that $g_{,v}(u, v)$ must also vanish. Then, differentiating (23) with respect to σ and comparing it with the substitution of $\delta \to \delta(\sigma)$ into the equation $g_{,v}(u, v) = 0$ we obtain $\delta'(\sigma) = -1$. Integrating this equation we obtain $\delta(\sigma) = -\sigma + 2c$, where c is a constant. Owing to the definition of δ and σ, the boundary of the domain is determined by a constant value of $u, u = c$. In other words, $(g[c, v])^{-1} = 0$ for all admissible values of v. This constant can be determined by taking the limit $v \to c$. At this point $\delta = A(\delta, \sigma) = 0$ and $\sigma = 2c$. In other words, the equation

$$S(2c) = 0 \tag{24}$$

provides the corresponding value of c.

Note now that when $g_{uu} = g[u, v]$ has a pole at $u = c$ $g_{vv} = g[v, u]$ also has a pole at $v = c$. Then, whenever (24) has a positive solution the range of variables u and v is the square $|u|, |v| < c$.

Let us examine now the solution of (24) for our four metric solutions. When $a > 0$ the boundary $r = 0$ separates domains with or without solutions. However, when $a < 0$ the corresponding boundary value is $r = r_0$, where $r_0 = -1$ for solution I and $r_0 = -50/33$ for solution II. Owing to the identities $r = -6a^2\lambda$ and $1 - r/r_0 = a^2 20\Lambda_6$, these boundaries separate domains with different curvature: $r > 0$ corresponds to AdS$_4$ and $r < 0$ to dS$_4$ while $r > r_0$ corresponds to AdS$_6$ while $r < r_0$ corresponds to dS$_6$. Thus, the boundaries are:

(i) Solution I, $a > 0$

The boundary, c, satisfies

$$-r(1 + 2ca)^2 + 1 + r = 0 \tag{25}$$

where $r = -\lambda/6a^2 > 0$. Then for $r > 0$ (AdS$_4$) the range of variables u and v is

$$|u|, |v| < \frac{1}{2a} \left(\sqrt{\frac{r+1}{r}} - 1 \right). \tag{26}$$

For $r < 0$ (dS$_4$) the range of $|u|$ and of $|v|$ is infinite.

(ii) Solution II, $a > 0$

(24) has the form

$$-r(1 + 2ca)^2 + 1 + \tfrac{33}{50}r + \tfrac{17}{50}r(1 + 2ca)^5 = 0. \tag{27}$$

No analytic solution of this fifth-order equation exists but at small positive values of r the range of u and v is infinite. Increasing r we obtain a critical value, above which the range becomes finite, because (27) has a positive solution. This critical value is obtained from the coincidence of the zero (27) with the zero of the derivative $S(\sigma)$. This happens at

$$r_{\min} = \frac{1}{w_{\min}^2 - \frac{33}{50} - \frac{17}{80}w_{\min}^5} \simeq 115.5, \tag{28}$$

where

$$w_{\min} = \left(\frac{20}{17} \right)^{1/3} = 1 + 2ac_{\min}. \tag{29}$$

If r increases, starting from $r_{\min}$ then the boundary value of $|u|$ and $|v|$ decreases from $c_{\min}(r_{\min}) = (1/2a)[(20/17)^{1/3} - 1] = \frac{1}{a}0.2095$ to $c_{\min}(\infty) = \frac{1}{a}0.0538$. For negative values of r (27) always has a positive root, so the range of variables is finite.

(iii) Solution I, $a < 0$

The largest possible physical range of $w = 1 + a\sigma$ is $0 < w < 1$. There is a solution of the equation $S(\sigma) = 0$ only if $r < -1$. This corresponds to dS$_6$ space. The solution of the $S(\sigma) = 0$ equation provides the following bounds for

$$|u|, |v| < \frac{1}{2|a|} \left(1 - \sqrt{\frac{|r| - 1}{|r|}} \right). \tag{30}$$

For $r > -1$ the range of u and v is bounded only by the condition $|u| + |v| < 1/|a|$ (i.e. finiteness of the conformal factor).

(iv) Solution II, $a < 0$

At $w = 1$, $S = 1$. At $w = 0$, $S = 1 + 33, r/50$. Therefore, if $r > r_0 = -50/33$ (AdS$_6$) there is no solution of the $S(\sigma) = 0$ equation (it is easy to see that there is no minimum of $S(\sigma)$ lower than 0). However, there is a solution of the equation $S(\sigma) = 0$ for $r < -50/33$ because $S(w = 0) < 0$. So $w \geqslant w_{\min} > 0$. These values of r correspond to dS$_6$ space ($\Lambda_6 > 0$). Just like in the case $a > 0$ there is no analytic solution for $w < w_{\min}$.

3. Small oscillations

Suppose the metric solutions we previously found, denoted $^0g_{ab}$, are modified by a small perturbation, such as

$$g_{ab}(\epsilon) = {}^0g_{ab} + h_{ab}. \tag{31}$$

As now it is standard in the literature, we will impose the TT (traceless, transverse) gauge conditions on h_{ab}. In addition we will impose axial gauge conditions, $h_{au} = h_{av} = 0$. Thus, the nonzero components of the TT tensor h are $h_{\mu\nu}$, where $\mu, \nu = t, x, y, z$. This gauge is also called the Randall–Sundrum gauge. The advantage of this gauge (in contrast to the TT conditions applied to the 'complete oscillations', $\Omega^2 h_{ab}$), as we will see below, is that for the graviton mode $h_{\mu\nu}$ is independent of u and v. Then, at least for $a > 0$ the conformal factor Ω^2 insures localization. In general, gravitational excitations are dependent on these variables, as well. They may or may not be localized to the intersection of the branes. In what follows, we will drop the superscript 0 from the background metric, ${}^0g_{ab}$.

The components of h satisfy the following wave equation:

$$\tfrac{1}{2}\nabla^A \nabla_A h + \tfrac{1}{5}\Lambda h = 0, \tag{32}$$

where $A = t, x, \ldots, u, v$ and ∇^A is a covariant derivative in the background metric, γ_{AB}. Equation (32) is separable. Multiplying by the conformal factor we obtain

$$\frac{g^{\mu\nu}}{2}\tilde{\nabla}_\mu \tilde{\nabla}_\nu h + \frac{1}{2}\nabla^{u_i}\nabla_{u_i} h + \frac{1}{5}\Lambda h = 0, \tag{33}$$

where the covariant derivatives $\tilde{\nabla}_\mu$ are taken in the four-dimensional metric $g_{\mu\nu}$ and u_i is a shorthand notation for u and v.

Now the first term of (33) contains only derivatives and metric components dependent on t, x, y and z, while the rest of the terms depend on u and v only. Using a factorized ansatz for the wavefunction and introducing the gravitational mass m we obtain the following two equations (here we assume a specific form for the four-dimensional metric: Minkowski with a warp factor $g_{yy} = g_{xx} = -g_{tt} = \omega^2 = e^{-2bz}$ for AdS$_4$ and $g_{ii} = \omega^2 = e^{-2bt}$, where i denotes a spatial coordinate, for dS$_4$)

$$m^2 h = g^{\mu\nu}\partial_\mu \partial_\nu h \pm \frac{7}{2}b\partial_\xi h - \frac{5}{3}\lambda h, \tag{34}$$

$$m^2 h = \left[\frac{\lambda\Omega^{-1}}{4a} - \frac{\Lambda_6\Omega}{8a} + \frac{5a\Omega}{4g(u,v)} + \frac{3a\Omega}{4g(v,u)} - \frac{\partial_v g(u,v)}{g(u,v)}\left(\frac{1}{g(u,v)} + \frac{1}{g(v,u)}\right)\right]\partial_u h$$
$$+ \frac{\partial_u^2 h}{g(u,v)} + (u \leftrightarrow v), \tag{35}$$

where the upper sign corresponds to dS$_4$ and the lower one to AdS$_4$. We use $\xi = z$ or t and $\lambda = \pm 3b^2$.

The solution of (34) is straightforward. It describes wave oscillations

$$h_1 \sim e^{i(k_x x + k_y y + k_z z - k_t t)}, \tag{36}$$

where $k_t = \sqrt{k_x^2 + k_y^2 + k_z^2 + m^2}$, provided the coordinates are much smaller than the curvature radius, $1/b$, otherwise the wave is modified, as we will discuss it below. In particular, the $m = 0$ mode always exists since $h_2 = \text{const}$ is a solution of (35). It describes a graviton, as long as the corresponding wavefunction is normalizable. In what follows, we will examine the solutions of (35) and their normalization.

The exact solution of (34) for dS$_4$, using an ansatz $h = e^{i(xk_x + yk_y + zk_z)}f[t]$, is

$$f[t] = e^{7bt/4}H_n^{(2)}(e^{bt}k/b) \tag{37}$$

where $H_n^{(2)} = J_n + iY_n$ is the Hankel function of the second kind, $n = i\sqrt{m^2/b^2 + 31/16}$ and $k = \sqrt{k_x^2 + k_y^2 + k_z^2}$. When one uses the asymptotic expansion for large order but fixed

argument to order (i.e. taking the limit $b \to 0$) [15] then we can write, after factoring out a diverging constant

$$f[t] \sim e^{-it\sqrt{k^2+m^2}+O(b/k)}. \tag{38}$$

As a final note, as we indicated earlier, we mention that (35) illuminates the special role of the seemingly arbitrarily chosen conformal factor, Ω. The ground-state wavefunction ($m = 0$ solution of (35)) is constant. Note however that the metric, and with that h was defined in (1) with a factor Ω^2 extracted. Thus the true ground-state wavefunction (provided it is normalizable) is exactly Ω^2 for every choice of our parameters.

3.1. Mass spectrum of excitations

In general, (35) is too complicated to be transformed to the volcano potential form, familiar in five-dimensional brane theories, and solved using standard methods for solving a one-dimensional Schrodinger equation. Consider, however that the coordinates $\sigma = u + v$ and $\delta = u - v$ are somewhat analogous to the radial and angular coordinates in a Schrödinger equation. Thus, one expects, that solutions dependent on the 'radial coordinate', σ, only well represent the spectrum of excitations. Among others, the graviton state, with a constant wavefunction, is clearly such a state.

Indeed, using (3) we can see that for functions, independent of δ (35) reduces to

$$-Sh'' + \text{sign}(\sigma)\left(\frac{1}{2a}\tilde{\lambda}\Omega^{-1} - \frac{1}{4a}\tilde{\Lambda}_6\Omega - a\Omega S\right)h' = m^2 h, \tag{39}$$

where function S is given in table 1. Note that the coefficients of (39) are independent of δ, just like in a Schrodinger equation with rotational invariant potential the equation for rotational invariant states is independent of the angles. The reason for this cancellation is that the explicit δ-dependence of (35) comes only through multipliers $A(\delta, \sigma)$, which in turn are associated with a differentiation with respect to δ.

Before we can bring (39) to a Schrödinger form we need to introduce a new variable, ω, defined by

$$\omega = \text{sign}(\sigma) \int_0^{|\sigma|} \frac{\mathrm{d}s}{\sqrt{S(s)}}. \tag{40}$$

The integration in (40) can be analytically performed in the case of solution I only. After the change of variables we can eliminate the first derivative from (39) and obtain a Schrödinger equation with a volcano potential. This is done by factoring out an appropriate function $\rho(\omega)$ from h. Finally, we obtain the eigenvalue equation with the volcano potential in the form

$$-\partial_\omega^2 h + V(\omega)h - \delta(\omega)\frac{1}{4a}[2\lambda - \Lambda_6 - 4a^2 + 2aS'(0)]h = m^2 h, \tag{41}$$

where

$$V(\omega) = -\frac{\lambda}{2} + \Omega^2 \frac{(\Lambda_6 - 2\lambda\Omega^{-2})^2}{64a^2 S} - \frac{1}{4}a^2 S\Omega^2 - \frac{\Omega}{8aS}S'(\Lambda_6 - 2\lambda\Omega^{-2}) + \frac{3S'^2}{16S} - \frac{S''}{4}. \tag{42}$$

We use the notation h for the eigenfunction, though strictly speaking the function h in (41) differs by a factor of ρ from that used in (39).

In Ω and S the variable σ is understood to be substituted by $\sigma(\omega)$, the solution of (40). The explicit form to the potential in terms of variable ω can only be given for solution I, but using (41) one can qualitatively analyze the spectrum for solution II, as well.

The range of variable ω and the behavior of the potential at the end of the range, ω_0, plays an essential role in the nature of the mass spectrum. Naturally, that range is dependent on

the choice of gauge, but the variable ω is the natural coordinate. Anyway, the mass spectrum itself is gauge invariant.

The $m = 0$ eigenfunction of (39), candidate for the graviton as long as it is normalizable, is a constant. Then the corresponding eigenfunction of (41) is proportional to $[\rho(\omega)]^{-1}$ and given by

$$
\begin{aligned}
h_0(\omega) &= \exp\left\{\frac{1}{8a}\int_0^\omega [\Omega(\Lambda_6 - 2\lambda\Omega^{-2}) + 4a^2\Omega S - 2a^2 S']\frac{\mathrm{d}\omega'}{\sqrt{S}}\right\} \\
&= \frac{w^{1/2}}{S^{1/4}}\exp\left\{\frac{1}{8a^2}\int_1^w (\Lambda_6 - 2\lambda w^2)\frac{\mathrm{d}w}{wS}\right\},
\end{aligned}
\tag{43}
$$

where $\Omega = 1/w$, S, and S' must be regarded as functions of ω. Here S' is the derivative of S with respect to $a\sigma$. Note now that h_0 has no nodes, so provided it is normalizable it is the graviton state and the ground state of the eigenvalue equation.

One can make a general comment concerning the mass spectrum. The potential depends on λ only through the dimensionless parameter r that characterizes the function $S(\sigma)$ completely. The remaining parameter a just sets the scale for the excitation spectrum. The value of r fixes the asymptotic curvature of the six- and four-dimensional spaces. Introducing the parameter $r_0 = -1$ for solution I and $r_0 = -50/33$ for solution II we have

- dS_6 and dS_4 for $r < r_0$;
- M_6 and dS_4 for $r = r_0$;
- AdS_6 and dS_4 for $0 > r > r_0$;
- AdS_6 and M_4 for $r = 0$;
- AdS_6 and AdS_4 for $r > 0$.

3.2. Normalization

The normalization condition is very simple in the Schrödinger type, 'ω-representation'. The norm is finite if the integral over the square of the wavefunction converges. It is sufficient to know the wavefunction at the endpoint of the range of ω, ω_0 (possibly infinity), as only the singular behavior at ω_0 can make the solution non-normalizable. Even when we are not able to solve the eigenvalue equation exactly we are able to determine the behavior of the solutions at ω_0. However, the true restriction on the eigenfunctions is stronger. We must consider only solutions that can be orthonormalized, otherwise the small oscillation cannot be expanded into modes of definite mass. The condition of orthogonalization is that the Wronskian, $h_1 h_2' - h_2 h_1'$ vanishes at $\omega = \omega_0$ [16]. If the combination $h_1 h_2'$ is infinite then this condition is certainly violated. This condition is particularly important in the case of inverse square potentials, which will often make their appearance below.

Every case in which the range of ω is finite the potential energy has the endpoint behavior $V \simeq -\frac{1}{4}(\omega_0 - \omega)^{-2}$. The only acceptable solution behaves (neglecting subleading terms of the potential) as a Bessel function, $\sqrt{x}J_0(mx)$, near $x = \omega_0 - \omega = 0$. The Wronskian vanishes for a pair of such solutions at different values of m and the solutions are orthogonal. The Wronskian for two singular solutions, which behave like $\sqrt{x}Y_0(m_i x)$, diverges logarithmically at $x = 0$ making these solutions unacceptable. Having a unique acceptable solution implies that imposing the boundary condition at $\omega = 0$ introduces the usual quantization condition and the spectrum is discrete. Then, owing to the fact that in terms of the original variables, u and v a zero mass solution (independent of u and v) exists, there is a physical graviton state.

Note that the normalizability and orthogonalizability of the solutions of the Schrodinger-like equation in variable ω does not insure the finiteness of the contribution of the corresponding small oscillations to quantities like the 'energy' defined by Mannheim [16]. That requires

Table 2. Solution I: curvature of space, range of ω, properties of the excitation spectrum and the range of weakly coupled states, as a function of parameter r.

r	Space	Graviton state	Level spacing	Weakly coupled
$r > 0$	AdS$_6$, AdS$_4$	Isolated	$\Delta m \sim (-\lambda)^{1/2}$	$m < (-\Lambda)^{1/2}$
$r = 0$	AdS$_6$, M$_4$	Marginal	Continuous spectrum	$m < (-\Lambda)^{1/2}$
$-1 < r < 0$	AdS$_6$, dS$_4$	Isolated	$\Delta m \sim (-\lambda)^{1/2}$	$m < (-\Lambda)^{1/2}$
$r = -1$	M$_6$, dS$_4$	Isolated	$\Delta m \sim (-\Lambda)^{1/2}$	None
$\leqslant -1$	dS$_6$, dS$_4$	Isolated	$\Delta m \sim (-\Lambda)^{1/2}$	None

Table 3. Solution II: curvature of space, range of ω, properties of the excitation spectrum and the range of weakly coupled states, as a function of parameter r.

r	Space	Graviton state	Level spacing	Weakly coupled
$r > r_0$	AdS$_6$, AdS$_4$	Isolated	$\Delta m \sim (-\Lambda)^{1/2}$	None
$r = r_0$	AdS$_6$, AdS$_4$	Isolated	Continuous spectrum	None
$r_0 > r > 0$	AdS$_6$, AdS$_4$	Isolated	$\Delta m \sim (-\lambda)^{1/2}$	$m < (-\Lambda)^{1/2}$
$r = 0$	AdS$_6$, M$_4$	Marginal	Continuous spectrum	$m < (-\Lambda)^{1/2}$
$-50/33 < r < 0$	AdS$_6$, dS$_4$	Isolated	$\Delta m \sim (-\lambda)^{1/2}$	$m < (-\Lambda)^{1/2}$
$r = -50/33$	M$_6$, dS$_4$	Isolated	$\Delta m \sim (-\Lambda)^{1/2}$	None
$\leqslant -50/33$	dS$_6$, dS$_4$	Isolated	$\Delta m \sim (-\Lambda)^{1/2}$	None

the convergence of integrals over the variables u and v of the covariant weight, $\sqrt{g}$ and the square of the small oscillation. This constraint come into focus for $a < 0$ solutions, for which, depending on r, the end of range of $w = 1 + a(|u| + |v|)$ may be $w = \Omega^{-1} = 0$, at which point $\int d\delta \sqrt{g} \sim w^{-5}$ has a singularity.

In tables 2 and 3 we summarized our results for both of our solutions for $a > 0$. The nature of the solutions changes as a function of $r = -\lambda/a^2$, among others the curvature of the six- and four-dimensional spaces, the question whether the graviton state is isolated, the order of level spacing of the spectrum, and the range of mass excitations with a weak coupling is displayed. The last two columns are important only for $\lambda \ll \Lambda$, i.e. $|r| \ll 1$. This is the range in which corrections to the short and long range behavior of gravity do not contradict to observations in our current universe.

3.3. Mass spectrum for $a < 0$

If $S(\sigma)$ has no zero in the interval $0 < w < 1$ (here $w = 1 - |a|\sigma$) then the solution is always unacceptable due to the divergence of the measure at $w = 0$. In all cases when a zero exists the parameter r is large. Though these solutions are mathematically acceptable, they could not be used for phenomenological purposes, for reasons discussed in the $a > 0$ case.

4. Conclusions

We solved the six-dimensional Einstein equations and Israel junction conditions in a space with mirror symmetry and exchange symmetry, $(Z_2)^3$, in the two extra dimensions, when the four-dimensional space at the intersection of two five-dimensional branes is warped. Due to their coordinate dependence, the junction conditions can only be solved if the gauges in the six- and five-dimensional actions are misaligned. We found two solutions, one with induced gravity terms on the branes, and other with an induced cosmological constant only. For each of these solutions we found two possible alternatives, having conformal factors that are increasing and decreasing, respectively, as the modulus of the fifth or sixth coordinate increases. We

also studied small perturbations around our solutions. Due to the complexity of the equations satisfied by these perturbations we restricted ourselves to those depending only on the sum of the extra coordinates. One of the oscillatory solutions of these equations is the massless graviton state, which is always localized at the intersection of the 4-branes. In addition there is an infinite set of solutions corresponding to gravitational excitations. The analytic form of the 'volcano potential' could only be derived for the solution with induced five-dimensional gravity, but the graviton state and the nature of the gravitational excitation spectrum could always be analyzed. We found that for increasing conformal factors the graviton state is not normalizable. This is similar to the situation found in five dimensions by Karch and Randall [6]. However, here the lowest lying normalizable state is massive, leading to an unacceptable short range gravitational force. For decreasing conformal factor, depending on the values of the appropriately normalized six-dimensional and four-dimensional cosmological constants, $\tilde{\Lambda}_6$ and $\tilde{\lambda}$, we obtained a variety of spectra. For most values of the cosmological constants we found an acceptable spectrum, with a zero mass graviton localized at the intersection of the two branes and a gravitational excitation spectrum that can be tuned not to lead to contradictions with experiments verifying Newton's law in the laboratory, provided $\tilde{\Lambda}_6 >> \tilde{\lambda}$. This work is an extension of similar investigations in five dimension [6–9] to six dimension. Previous studies of a six-dimensional model with similar symmetries allowed only for M_4 space at the intersection of the branes [5]. The model we studied requires the input of the value of the four-dimensional cosmological constant by hand. We find this esthetically not very satisfactory. This blemish could in principle be circumvented by the addition of a global scalar, or several scalars, like quintessence, which could possibly generate a cosmological constant which may even depend on time. We intend to return to this problem in a future publication. We are somewhat perplexed by the existence of two acceptable solutions at given values of the input parameters (cosmological constants). We have not been able to find a physical principle which would allow us to discard one of the solutions yet. One possibility is to investigate observable post-Newtonian effects. The models under discussion have two separate gravity actions in six and five dimensions. Therefore, just like the Dvali–Gabadadze–Porrati (DGP) model [12], they could suffer from the van Dam–Veltman–Zakharov [13, 14] type of discontinuity and could be severely constrained. For more details we refer the reader to the following papers [17–20].

Acknowledgments

Appendix A. Solution of the global equations

The two Einstein equations that we use to find the solution are then

$$
\mathcal{G}_{uu} = g_{uu} \left[\frac{10a^2}{g(u,v)} + \frac{10a^2}{g(v,u)} - 2\tilde{\lambda}(1 + a(u+v))^2 \right.
$$
$$
\left. - (1 + a(u+v)) \frac{2a}{g(v,u)} \left(\frac{g_{,u}(v,u)}{g(u,v)} - \frac{g_{,v}(v,u)}{g(v,u)} \right) \right] = -\tilde{\Lambda}_6 g_{uu}, \tag{A.1}
$$

$$
\mathcal{G}_{uv} = -\frac{2a}{1 + a(u+v)} \left(\frac{g_{,v}(u,v)}{g(u,v)} + \frac{g_{,u}(v,u)}{g(v,u)} \right) = 0, \tag{A.2}
$$

where $\tilde{\lambda} = 8\pi G_4\lambda$. Substituting $\frac{g_{,u}(v,u)}{g(v,u)}$ from (A.2) into (A.1) we obtain a first-order differential equation for the symmetric function

$$F(u, v) = \frac{1}{2g(u, v)} + \frac{1}{2g(v, u)} \tag{A.3}$$

$$20a^2 F(u, v) - 2\lambda(1 + a(u + v))^2 + \tilde{\Lambda}_6 - (1 + a(u + v))4a F_{,v}(u, v) = 0. \tag{A.4}$$

First, these equations imply that the derivative $F_{,v}(u, v)$ is also symmetric. That is possible only if $F(u, v)$ is a function of $\sigma = u + v$ only. Keeping this in mind it has an analytic solution

$$F(u, v) \equiv S(\sigma) = \frac{1}{a^2}\left[(1 + a\sigma)^2\frac{\lambda}{6} - \frac{\tilde{\Lambda}_6}{20} + C(1 + a\sigma)^5\right], \tag{A.5}$$

where C is an integration constant.

Now defining $\delta = u - v$ and using (A.3) we can write $g(u, v)$ in terms of an odd function of δ, $A(\delta, \sigma)$ as

$$g(u, v) = \frac{1}{S(\sigma) + A(\delta, \sigma)}. \tag{A.6}$$

Inserting (A.6) into (A.2) we obtain a differential equation for $A(\delta, \sigma)$

$$A(\delta, \sigma)A_{,\sigma}(\delta, \sigma) + S(\sigma)[A_{,\delta}(\delta, \sigma) - S'(\sigma)] = 0. \tag{A.7}$$

We have not been able to give a solution of (A.7) in closed form, but the terms of its Taylor series in the variable δ can be readily calculated. We will write

$$A(\delta, \sigma) = \sum_{k=0}^{\infty} \delta^{2k+1}\alpha_k(\sigma). \tag{A.8}$$

Then by substituting (A.8) into (A.7) we obtain

$$\alpha_0(\sigma) = S'(\sigma), \tag{A.9}$$

and the recursion relation

$$\alpha_k(\sigma) = -\frac{1}{(2k + 1)S(\sigma)}\sum_{m=0}^{k-1}\alpha_{k-m-1}(\sigma)\alpha'_m(\sigma). \tag{A.10}$$

Using the recursion relation we can easily generate the coefficients $\alpha_k(\sigma)$ in arbitrary order. We obtain

$$\alpha_1(\sigma) = -\frac{S'(\sigma)S''(\sigma)}{3S(\sigma)}.$$

$$\alpha_2(\sigma) = \frac{S'(\sigma)(-S'(\sigma)^2 S''(\sigma) + 2S(\sigma)S''(\sigma)^2 + S(\sigma)S'(\sigma)S^{(3)}(\sigma))}{15S(\sigma)^3},$$

$$\alpha_3(\sigma) = -\frac{S'(\sigma)}{315S(\sigma)^5}[9S'(\sigma)^4 S''(\sigma) + 17S(\sigma)^2 S''(\sigma)^3 - 9S(\sigma)S'(\sigma)^3 S^{(3)}(\sigma)$$
$$+ 26S(\sigma)^2 S'(\sigma)S''(\sigma)S^{(3)}(\sigma) + S(\sigma)S'(\sigma)^2(-29S''(\sigma)^2 + 3S(\sigma)S^{(4)}(\sigma))],$$
$$\cdots$$

For the special case of $\Lambda_6 = 0$, i.e. assuming that the six-dimensional cosmological constant vanishes, still we get a curved space on the (3+1)-dimensional world. In fact, at least in solution 1, $A(\delta, \sigma)$ simplifies considerably. We will consider this case next.

First, $S(\sigma) = w^2$, where we introduced the notation $w = 1 + a\sigma$. If we also introduce the notation

$$z = 1 - \frac{a\delta}{1 + a\sigma} \tag{A.11}$$

and we regard A as a function of w and z, then A can be written in the form

$$A = w^2 h(z), \tag{A.12}$$

where $h(z)$ satisfies the ordinary differential equation

$$2h^2 + hh'(1 - z) - h' = 2. \tag{A.13}$$

We know the expansion of $h(z)$ around $z = 1$ already. It is quite interesting to investigate $h(z)$ around $z = 0$, as well. The exact point, $z = 0$ is not physical as $a\delta < 1 + a\sigma$, but at large u and small v (near the u axis) z approaches 0. Numerical solution of this equation shows that $h(0) = 1$ and $h'(1) = 0$. Of course we know that $h(1) = 0$. Then we can write $h(z) = 1 - k(z)$ and expand (A.13) in k, keeping linear terms only. Simple calculation shows that (A.13) reduces to $-4k + zk' = 0$ leading to $h(z) = 1 - cz^4 + O(z^7)$ (here we also calculated the next to leading term of k). This form is consistent with the expansion of (A.13) in a power series of k. The constant c can be identified from the numerical calculation as $c = 1/4$.

The case $\Lambda_6 \neq 0$ can also be investigated analytically near $z = 0$ (near the axes, far from the origin). Using the ansatz

$$A(\delta, \sigma) = S(w) - h(z)f(w) \tag{A.14}$$

(A.7) can be expanded to retain the first power of h only. One obtains a separable equation

$$h(z)[S(w)f'(w) + f(w)S'(w)] - S(w)f(w)h'(z)z = 0. \tag{A.15}$$

The solution of this equation is

$$f(w) = \frac{w^n}{S(w)}, \qquad h(z) = cz^n. \tag{A.16}$$

Note that in the special case $S = w^2$ the integration constant $n = 4$.

Appendix B. Junction conditions with gauge transformations

When we perform (12) the components of the metric tensor, g_{uu}, g_{vv} and g_{uv} change in the following way (the timelike and ordinary spacelike components are unchanged aside from the replacements of the variables)

$$g_{uu} = \frac{g(u, v)}{1 + a(u + v))^2} \rightarrow \tilde{g}_{uu} = R[\tilde{g}(u, v)[h_{,v}(v, u)]^2 + \tilde{g}(v, u)[h_{,u}(v, u)]^2],$$

$$g_{vv} = \frac{g(v, u)}{(1 + a(u + v))^2} \rightarrow \tilde{g}_{vv} = R[\tilde{g}(u, v)[h_{,u}(u, v)]^2 + \tilde{g}(v, u)[h_{,v}(u, v)]^2], \tag{B.1}$$

$$g_{uv} = 0 \rightarrow \tilde{g}_{uv} = -R[\tilde{g}(v, u)h_{,u}(u, v)h_{,u}(u, v) + \tilde{g}(u, v)h_{,v}(v, u)h_{,v}(u, v)],$$

where we use the notation $\tilde{g}(u, v) = g(h(u, v), h(v, u))$, $\tilde{g}(v, u) = g(h(v, u), h(u, v))$ and where

$$R = \frac{1}{[h_{,u}(u, v)h_{,v}(v, u) - h_{,v}(u, v)h_{,u}(v, u)]^2\{1 + a[h(u, v) + h(v, u)]\}^2}. \tag{B.2}$$

Now note that the components of the gauge transformed metric tensor at $u = v = 0$ are given by $\tilde{g}_{uu}(0, 0) = \tilde{g}_{vv}(0, 0) = g(0, 0)[h_{,u}(u, v)|_{u=v=0}]^2$. Since normalization requires that these quantities are equal to 1 we have imposed the condition $h_{,u}(u, v)|_{u=v=0} = 1$ on (12).

In this appendix we will prove that the gauge transformed junction contributions of the six-dimensional theory can be canceled by the five-dimensional curvature scalar and cosmological constant terms. First we need to find the gauge transformed form of the junction contributions (9). We will concentrate only on the contribution at the $v = 0$ brane as once we solved the junction conditions at the $v = 0$ brane, the junction conditions at the $u = 0$ brane are automatically satisfied considering the symmetry of gauge transformations (12). As a reminder, we must take linear combination of the $\Delta \tilde{R}^{(6)}_{uu}$, $\Delta \tilde{R}^{(6)}_{vv}$ and $\Delta \tilde{R}^{(6)}_{uv}$ contributions, such that they correspond to functional derivatives with respect to the untransformed metric components. These combinations are

$$
\begin{aligned}
\Delta_v R^{(6)}_{\mu\mu} &= -g_{\mu\mu} \frac{(1+au)^2}{1+ah} \delta(v)[h_{,v}(v,u)|_{v=0}]^3 \frac{8ag(h,0) - (1+ah)g_{,v}(h,v)|_{v=0}}{g(h,0)g(0,h)}, \\
\Delta_v R^{(6)}_{uu} &= -g_{uu} \frac{(1+au)^2}{1+ah} \delta(v)[h_{,v}(v,u)|_{v=0}]^3 \frac{8ag(h,0)}{g(0,h)},
\end{aligned}
\tag{B.3}
$$

where we used the abbreviated notation $h = h(u,0)$.

Now we are ready to substitute (11) and (B.3) into junction condition (13). Note that the two conditions ($J_1(u) = 0$ and $J_2(u) = 0$) at $AB = \mu\mu$ and at $AB = uu$ contain two independent functions: $k(u) = h(u,0)$ and $f(u) = h_{,v}(v,u)|_{v=0}$. These two functions are unconstrained aside from the following conditions: $k(0) = 0, k'(0) = 1, f(0) = 1$. $f(u)$ appears in a very simple manner in the two junction conditions as shown by (B.3). We can easily eliminate this function from the two junction conditions by taking the ratio of the two junction conditions. The resulting condition (call it $F(k(u), u) = 0$) is a transcendental function of $k(u)$ (not a differential equation) and, in principle, can be solved for it. It has the form

$$
\frac{8ag(k,0) - (1+ak)g_{,v}(k,v)|_{v=0}}{8ag(k,0)} = \frac{\lambda(1+au)^2 - \tilde{\Lambda}_5 - \frac{6a^2}{g(u,0)} - \frac{3ag_{,u}(u,0)(1+au)}{2[g[u,0]]^2}}{2\lambda(1+au)^2 - 8\pi G_5\Lambda_5 - \frac{6a^2}{g(u,0)}}.
\tag{B.4}
$$

Once we solved for $k(u)$ we can calculate $f(u)$ as well. All this can be done provided the three constraints on $k(u)$ and $f(u)$ are satisfied. Note that due to these three constraints the junction conditions $J_1(0) = 0$, $J_2(0) = 0$ and $\mathrm{d}F(h(u), u)/\mathrm{d}u|_{u=0} = 0$ become independent of the form of the gauge transformations. All other (higher) derivatives of the junction conditions contain unconstrained derivatives of the gauge transformation function. These provide equations for those unconstrained derivatives. We can conclude that the junction conditions can be solved for the gauge transformations provided three constraints among the constants of the theory are satisfied. The conditions $J_1(0) = 0$ and $J_2(0) = 0$ were given earlier in (14) and (15), while the condition $\mathrm{d}F(f_1(u), u)/\mathrm{d}u|_{u=0} = 0$ was given in (21).

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